Exercise 11

Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

and then solving a pair of simultaneous equations in x and y.

Suggestion: Use the fact that no real number x satisfies the given equation to show that $y \neq 0$.

$$\textit{Ans. } z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$$

Solution

Multiply (x, y) and (x, y) together and add (x, y) and $(1, 0)$ together.

$$\begin{aligned}(x^2 - y^2, xy + xy) + (x + 1, y + 0) &= (0, 0)\\ (x^2 - y^2, 2xy) + (x + 1, y) &= (0, 0)\\ (x^2 - y^2 + x + 1, 2xy + y) &= (0, 0)\end{aligned}$$

In order for equality to hold, the real and imaginary parts of both sides must be equal.

$$\begin{aligned}x^2 - y^2 + x + 1 &= 0\\ 2xy + y &= 0\end{aligned}$$

As a result, the complex quadratic equation has been converted to a system of two equations for two real unknowns, x and y. Notice that the second equation is satisfied if $y = 0$ or $x = -1/2$. If $y = 0$, then the first equation becomes

$$x^2 + x + 1 = 0,$$

which has no real solution. If $x = -1/2$, then the first equation becomes

$$\begin{aligned}\frac{1}{4} - y^2 - \frac{1}{2} + 1 &= 0\\ y^2 &= \frac{3}{4}\\ y &= \pm\frac{\sqrt{3}}{2}.\end{aligned}$$

Therefore, the two solutions to the complex quadratic equation are

$$z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right).$$